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LIE CENTRE-BY-METABELIAN GROUP ALGEBRAS IN EVEN CHARACTERISTIC, II

BY

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ABSTRACT

We complete the classification of the Lie centre-by-metabelian group algebras over arbitrary fields by solving the case of characteristic 2.

The classification of the Lie centre-by-metabelian group algebras in characteristic p was started by Sharma and Srivastava in [12], and continued by Külshammer and Sharma in [4], and by Sahai and Srivastava in [10]. A. Bovdi remarked in his survey article [1] that the case $p = 2$ was yet to be resolved. The present paper is the second of two (cf. [9]) devoted to this task, with the following result:

THEOREM: *Let G be a group, and let $\mathbb{F}$ be a field of characteristic 2. Then $\mathbb{F}G$ is Lie centre-by-metabelian, if and only if one of the following conditions is satisfied:*

(i) $|G'|$ *divides* 4.
(ii) G' *is central and elementary abelian of order* 8.
(iii) G *acts by element inversion on* $G' \cong Z_2 \times Z_4$, *and* $\mathcal{C}_G(G')' \subseteq \Phi(G')$.
(iv) G *contains an abelian subgroup A of index* 2.

The following observation sheds some light on condition (iv):

LEMMA 0.2: *If a group G has an abelian subgroup A of index 2, then G acts on G' by element inversion. In particular, $\mathcal{C}_G(G') \in \{G, A\}$, and*

$$\exp(G') \leq 2 \iff \mathrm{cl}(G) \leq 2 \iff \mathcal{C}_G(G') = G \iff \mathcal{C}_G(G') \neq A.$$

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Proof: Let $g \in G$, $a \in A$. Then $G' = (g, A)$, $g^2 \in A$, and $1 = (g^2, a) = {}^{g}(g,a)(g,a)$. ∎

In [9], we already established the "if"-direction of Theorem 1. The proof of the "only-if"-direction falls into two parts: The first step is to show it for nilpotent groups of class 2, and for groups whose commutator subgroups have exponent 2 (which turns out to be the same here). This has been done in [9]. The objective of the present paper is the second step, namely to prove the "only-if"-direction for groups that act nontrivially on their commutator subgroups. In many cases, we will have to show that this action is in fact element inversion. So in section 1, we first prove two general lemmata which already yield some preliminary restrictions on how G may act on G', if $\mathbb{F}G$ is Lie centre-by-metabelian. We then describe an algorithm that computes all actions that comply with these restrictions in the case $|G'| \in \{8, 16\}$. Since it turns out that there are only few of these, we examine them one by one in sections 2 and 3. Thus prepared, we finally extend the proof of Theorem 1 to arbitrary groups in section 4.

The notation used in the text is standard (see also [9]); the base field always is denoted by $\mathbb{F}$, and always has characteristic 2.

1. Group actions and algorithmic reductions

LEMMA 1.1: *Let P be a normal finite 2-subgroup of a group G, such that $G/\mathcal{C}_G(P)$ is not a 2-group. If $\mathbb{F}G$ is Lie centre-by-metabelian, then*

(i) *there is a Hall $2'$-subgroup $S/\mathcal{C}_G(P)$ of $G/\mathcal{C}_G(P)$ of order 3,*

(ii) *$P = \mathcal{C}_P(S) \times (S, P)$ is abelian with $(S, P) \cong V_4$.*

Proof: Set $C := \mathcal{C}_G(P)$. Since P is finite, G/C is finite. By [7], G is solvable. Therefore G/C contains a (nontrivial) Hall $2'$-subgroup S/C [3, Hauptsatz VI.1.8].

By Burnside [2, theorem 5.1.4], the action of S/C on the elementary abelian group $\bar{P} := P/\Phi(P)$ is nontrivial and faithful. With $\bar{G} := G/\Phi(P)$, $\bar{S} := S\Phi(P)/\Phi(P)$, $\bar{C} := C\Phi(P)/\Phi(P)$, also $\bar{S}/\bar{C} \cong S/C$ acts nontrivially and faithfully on $\bar{P}$.

Applying [9, lemma 3.1] to $\bar{G}$ yields $|\bar{S} : \bar{C}| = 3$, $\bar{P} = (S, \bar{P}) \times \mathcal{C}_{\bar{P}}(S)$, and $|(S, \bar{P})| = 4$. In particular, $|S : C| = 3$; this shows (i). We write $S/C = \langle aC \rangle$.

To prove (ii), we first study the case $P = (S, P)$. We claim that then $P \cong V_4$. First note that $\bar{P} = P/\Phi(P) = (S, P)/\Phi(P) = (S, P/\Phi(P)) = (S, \bar{P})$ has order 4. Hence we may write $P/\Phi(P) = \langle x\Phi(P), y\Phi(P) \rangle$ with some elements

$x, y \in P$, where ${}^a x = y$. Then $P = \langle x, y\rangle$, and $x^2 \equiv y^2 \equiv (x,y) \equiv 1$, ${}^a y \equiv xy \pmod{\Phi(P)}$. Then

$$\begin{aligned}\rho :=&[x + {}^a x, a + {}^x a] = [x + y, a + xy^{-1}a] = [(1 + xy^{-1})y, (1 + xy^{-1})a]\\ =&(1 + xy^{-1})(y(1 + xy^{-1})a + a(1 + xy^{-1})y)\\ =&(1 + xy^{-1})(y + {}^y x + {}^a y + y)a = ({}^y x + {}^a y + x^2y^{-1} + xy^{-1}\,{}^a y)a,\end{aligned}$$

and

$$\begin{aligned}0 =&[x, \rho]a^{-1}\\ =&x({}^y x + {}^a y + x^2y^{-1} + xy^{-1}\,{}^a y)aa^{-1} + ({}^y x + {}^a y + x^2y^{-1} + xy^{-1}\,{}^a y)axa^{-1}\\ =&x\,{}^y x + x\,{}^a y + x^3y^{-1} + x^2y^{-1}\,{}^a y + yx + {}^a yy + x^2 + xy^{-1}\,{}^a yy \in \mathbb{F}P.\end{aligned}$$

Splitting the last sum w.r.t. the partition of P into cosets of $\Phi(P)$, we obtain

$$\begin{aligned}&0 = x\,{}^y x + x^2 \in \mathbb{F}[\Phi(P)], &&0 = x^2y^{-1}\,{}^a y + {}^a yy \in \mathbb{F}[x\Phi(P)],\\ &0 = x\,{}^a y + xy^{-1}\,{}^a yy \in \mathbb{F}[y\Phi(P)], &&0 = x^3y^{-1} + yx \in \mathbb{F}[xy\Phi(P)].\end{aligned}$$

The first equation implies that ${}^y x = x$, i.e. $P = \langle x, y\rangle$ is abelian. The last one shows that $x^2 = y^2$. Hence $\Phi(P) = \langle x^2, y^2\rangle$ is cyclic.

Assume that $\Phi(P) \neq 1$. Then $|\Phi(P) : \Phi(\Phi P)| = 2$. Since $\Phi(\Phi P) \trianglelefteq G$, we may replace G by $G/\Phi(\Phi P)$ if necessary, and assume that $|\Phi(P)| = 2$. Then $|P| = 8$, i.e. $P \cong Z_2 \times Z_4$. Then P contains four elements of order 4, and each automorphism that fixes one of them also fixes its inverse. Hence P has no automorphism of order 3, contradiction. Therefore $\Phi(P) = 1$, and thus $P \cong V_4$, as desired.

We now consider the case that $M := (S, P) < P$. By [5, 7.12], we have $P = MQ$, where $Q := C_P(S)$, and $(S, M) = (S, S, P) = (S, P) = M$. Therefore we are in a similar situation as in the preceding case (with M instead of P, and SM instead of G). Applying its result, we obtain $M \cong V_4$. Since S acts on M by cyclic permutation of the three nontrivial elements, we have $Q \cap M = 1$. Thus $P = M \rtimes Q$, and so $(M, Q) \in M \cap P' \subseteq M \cap \Phi(P)$. But $M \cong V_4 \cong M\Phi(P)/\Phi(P) \cong M/M \cap \Phi(P)$, i.e. $M \cap \Phi(P) = 1$. Consequently $(M, Q) = 1$, and $P = M \times Q$.

It remains to show that Q is abelian. We assume that $Q' \neq 1$ and set $U := \langle a, P\rangle$. Then $U' = (a, P)P' = (a, M)Q' = M \times Q'$. Since P is a finite 2-group, there exists a normal subgroup R of P with $R \subseteq Q'$ and $|Q' : R| = 2$ by [3, Satz III.7.2]. Since $R \subseteq Q$ is centralized by a, it is even normal in U. Then U/R is a not nilpotent, and $(U/R)' \cong M \times Q'/R \cong Z_2 \times Z_2 \times Z_2$, i.e. U/R is a counterexample to Theorem 1. But Theorem 1 has been proved in [9] for groups H with $\exp(H') = 2$. Contradiction. ∎

LEMMA 1.2: *Let G be a group such that $|G'| \not| 8$. Suppose that M is a subgroup of index 2 in G with $|M'| = 2$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: Note that $M' \trianglelefteq G$, hence $M' \subseteq \mathcal{Z}(G)$. We write $M' = \langle z\rangle$ and $G = \langle g, M\rangle$. Then $G' = (g, M)M'$.

We define maps τ: $(M \setminus \mathcal{Z}(M)) \times M \times M \to \mathbb{F}[G']$,

$$(b, c, d) \mapsto (1+z)(1+(g, c^{-1}))(1+(g, d))(1+(g, b)),$$

and φ: $M \to G'/M'$, $a \mapsto (g, a)M'$. Then φ is surjective; it even is an epimorphism since for all $a, b \in M$ we have $\varphi(ab) = (g, ab)M' = (g, a)\,{}^{a}(g, b)M' = (g, a)M'(a, g, b)(g, b) = (g, a)(g, b)M' = \varphi(a)\varphi(b)$. We are going to show that $\tau \neq 0$.

Suppose first that there are elements of G' whose order is not a power of 2. Then also G'/M' contains such elements. Then there clearly is also an element $b \in M \setminus \mathcal{Z}(M)$ such that $|\langle\varphi(b)\rangle|$ is not a power of 2; in particular, $(g, b)^4 \notin \langle z\rangle$. Then $(g, b^2) \equiv (g, b)^2 \pmod{\langle z\rangle}$, hence

$$\tau(b, b^{-2}, b) = (1+z)(1+(g, b^2))(1+(g, b))^2 = (1+z)(1+(g, b)^4) \neq 0.$$

Suppose now that every element of G' is a 2-element. Then $|G'| \geq 16$, i.e. $|G'/M'| \geq 8$. Then there also is a finite (2-)subgroup H/M' of G'/M' with $|H/M'| \geq 8$. We obviously may choose elements $c, d \in M$ such that $1 \lhd \langle\varphi(c)\rangle \lhd \langle\varphi(c), \varphi(d)\rangle \lhd H/M'$. Then $M_1 := \varphi^{-1}(\langle\varphi(c), \varphi(d)\rangle) < M$. Since also $\mathcal{Z}(M) < M$, there is an element $b \in M \setminus (M_1 \cup \mathcal{Z}(M)) \neq \emptyset$. Hence $1 < \langle\varphi(c)\rangle < \langle\varphi(c), \varphi(d)\rangle < \langle\varphi(c), \varphi(d), \varphi(b)\rangle$, i.e. $1 < \langle z\rangle < \langle z, (g, c)\rangle < \langle z, (g, c), (g, d)\rangle < \langle z, (g, c), (g, d), (g, b)\rangle$. Then $\tau(b, c^{-1}, d)$ does not vanish, since the summand 1 in its direct expansion cannot be cancelled.

In any case, there is a triple $(b, c, d) \in (M \setminus \mathcal{Z}(M)) \times M \times M$ such that $\tau(b, c, d) \neq 0$. Choose an element $a \in M \setminus C_M(b) \neq \emptyset$, then $(a, b) = z$, and

$$\begin{aligned}(\mathbb{F}G)'' \ni [b + {}^{a}b, g + {}^{c}g] &= [(1+z)b, (1+(c, g))g] \\ &= (1+z)(1+(c, g))[b, g] = \underbrace{(1+z)(1+(c, g))(1+(g, b))b}_{=:\sigma\in\mathbb{F}M}\,g.\end{aligned}$$

(Note that z is central and (c, g) commutes with b modulo $\langle z\rangle$.) Now $M/\langle z\rangle$ is abelian, in particular $[d, \sigma] = 0$. Then

$$\begin{aligned}0 \neq b\tau(b, c, d)dg &= (1+z)(1+(c, g))(1+(g, b))b(1+(g, d))dg \\ &= \sigma[d, g] = [d, \sigma g] \in [\mathbb{F}G, (\mathbb{F}G)''].\end{aligned}$$ ∎

Remark 1.3: Assume that G is a counterexample to Theorem 1, and that $|G'| \in \{8,16\}$. Set $H := G'$, $C := C_G(H)$, $A := G/C$, $I := H/(C\cap H) = H/\mathcal{Z}(H)$. Note that A and I may be embedded into $\mathrm{Aut}(H)$, using the group monomorphisms defined by the actions of A, resp. I, on H, where the image of I in $\mathrm{Aut}(H)$ is $\mathrm{Inn}(H)$. The canonical isomorphism $A' = HC/C \to I$ is compatible with these monomorphisms, hence A' is mapped onto $\mathrm{Inn}(H)$ as well.

Now H is not elementary abelian, and $A \neq 1$. (Recall that those are among the cases already discussed in [9], since $A = 1 \iff \mathrm{cl}(G) \leq 2$.) Furthermore, $\mathbb{F}G$ is Lie centre-by-metabelian, so Lemma 1.1 implies that:

(1) If H is nonabelian, then A is a 2-group.

(2) If H is abelian and A is not a 2-group, then A is a $\{2,3\}$-group such that $|S| = 3$ and $(S,H) \cong V_4$ for any Sylow 3-subgroup S of A.

If $|H| = 16$, we may argue by induction and assume that there are no counterexamples with commutator subgroups of order 8. So for all $N \trianglelefteq G$ with $N \subseteq H$ and $|N| = 2$, G/N is not a counterexample. Hence one of the following holds:

(3) $H/N \cong Z_2 \times Z_2 \times Z_2$, and A acts trivially on H/N, i.e. $(A,H) \subseteq N$.

(4) $H/N \cong Z_2 \times Z_4$, and A acts by element inversion on H/N.

(5) G/N contains an abelian subgroup of index 2.

But Lemma 1.2 shows that in case (5), G also contains an abelian subgroup of index 2, hence it is not a counterexample. So we may dismiss this case.

We are now able to describe an algorithm that, given the isomorphism type of H, computes all possibilities for A (as a subgroup of $\mathrm{Aut}(H)$ up to conjugacy):

- Check if H is elementary abelian. If so, stop, otherwise proceed.
- Compute (the conjugacy classes of) the subgroups of $\mathrm{Aut}(H)$.
- Throw away the trivial subgroup.
- Throw away all subgroups A with $A' \neq \mathrm{Inn}(H)$.
- Throw away the subgroups A that do not comply with either (1) or (2).
- If $|H| = 16$, let A loop over all subgroups that have survived so far. Compute all A-invariant subgroups N of H of order 2. If at least one of these N satisfies neither (3) nor (4), delete A from the list.

A computer program implementing this algorithm is described in [8, appendix C]. It is designed for the computer algebra system GAP [11], and may be downloaded from the authors web pages (filename actions.g). The following is a list of all groups H of orders 8 and 16, along with the results of the above algorithm:

catalogue	H	$\mathrm{Aut}(H)$	# of subgroups A
1	Z_8	V_4	4
2	$Z_2 \times Z_4$	D_8	6
3	D_8	D_8	0
4	Q_8	S_4	0
5	$Z_2 \times Z_2 \times Z_2$	$\mathrm{GL}(3,2)$	0
6	Z_{16}	$Z_2 \times Z_4$	0
7	$Z_4 \times Z_4$	(96)	1
8	(16)	(32)	0
9	(16)	(32)	0
10	$Z_2 \times Z_8$	$Z_2 \times D_8$	5
11	(16)	$Z_2 \times D_8$	0
12	D_{16}	(32)	0
13	Quasi-D_{16}	$Z_2 \times D_8$	0
14	Q_{16}	(32)	0
15	$V_4 \times Z_4$	(192)	3
16	$Z_2 \times D_8$	(64)	0
17	$Z_2 \times Q_8$	(192)	0
18	$Z_4 \mathrm{Y} D_8$	$Z_2 \times S_4$	0
19	$Z_2 \times Z_2 \times Z_2 \times Z_2$	$\mathrm{GL}(4,2)$	0

The first column indexes H as it appears in GAP's 2-group catalogue [6], the next two columns give the names for H and $\mathrm{Aut}(H)$ (resp. their orders in case they do not have a proper name), and the last column gives the number of (conjugacy classes of) subgroups A the algorithm has computed.

2. Commutator subgroups of order 8

In this section, we are going to verify Theorem 1 for all groups G with $|G'| = 8$. For this, it suffices to examine the $4+6 = 10$ cases listed in the upper half of the table in 1.3. In particular, we may assume that $G' \cong Z_8$ or $G' \cong Z_2 \times Z_4$.

Remark 2.1: Let G be a group with $G' = \langle x \rangle \cong Z_8$. Then $\mathrm{Aut}(G') = \langle \alpha, \beta \rangle \cong V_4$, where $\alpha\colon x \mapsto x^{-1}$, $\beta\colon x \mapsto x^3$. ∎

The four possibilities for the action of G on G' mentioned in the table in 1.3 are $\langle \alpha, \beta \rangle$, $\langle \alpha \rangle$, $\langle \beta \rangle$, $\langle \alpha\beta \rangle$, i.e. all nontrivial subgroups of $\mathrm{Aut}(G')$.

As usual, we set $C := C_G(G')$, and study the monomorphism $\varphi\colon G/C \hookrightarrow \mathrm{Aut}(G')$ that stems from the action of G on G'. We are going to show that if $\mathbb{F}G$

is Lie centre-by-metabelian, then the image of φ is $\langle\alpha\rangle$, and C is abelian (i.e. G satisfies condition (iv) of Theorem 1). This will be done in Lemmata 2.2–2.4.

LEMMA 2.2: *Let the notation be as in 2.1, and assume that the image of φ is either $\langle\beta\rangle$ or $\langle\alpha\beta\rangle$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: Observe that $\alpha\beta\colon G' \to G'$, $x \mapsto x^5$. Hence, there is an exponent $i \in \{3,5\}$ and an element $b \in G$ such that ${}^b x = x^i$. Then $G/C = \langle bC\rangle$, and $G' = (b,C)C' = (b,C)$, since $C' \subseteq \mathcal{Z}(G) \cap G' < G'$, i.e. $C' \subseteq \Phi(G')$.

Consequently, $(b,.)\colon C \to G' = \langle x\rangle$ is an epimorphism, so there is an element $c \in C$ such that $(b,c) = x$. Then

$$\begin{aligned}\tau :=&[[b,c],[b,cb^{-1}]] = [(1+x)cb,(1+x)c] = (1+x)c\,(b(1+x)c + (1+x)cb)\\ =&(1+x)c\left((1+x^i)x + (1+x)\right)cb = (1+x)(1+x^{i+1})c^2b,\end{aligned}$$

and

$$\begin{aligned}[c,\tau] = (1+x)(1+x^{i+1})c^2[c,b] &= (1+x)(1+x^{i+1})c^2(1+x)cb\\ &= (1+x^{i+1})(1+x^2)c^3b.\end{aligned}$$

It is easy to see that this is nonzero for any choice of $i \in \{3,5\}$. ∎

LEMMA 2.3: *Let the notation be as in 2.1, and assume that the image of φ is $\langle\alpha,\beta\rangle$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: Choose $a,b \in G$ with ${}^a x = x^{-1}$, ${}^b x = x^3$. Then ${}^{ab} x = x^5$, and $G/C = \langle aC, bC\rangle = \langle abC, bC\rangle$. From $C' \subseteq \mathcal{Z}(G) \cap G' = \mathcal{C}_{G'}(a) \cap \mathcal{C}_{G'}(b) = \langle x^4\rangle \subseteq \Phi(G')$, it follows that $G' = \langle(ab,b)\rangle\,(ab,C)(b,C)C' = \langle(a,b)\rangle\,(ab,C)(b,C)$. Since G' is cyclic, we have $G' = \langle(a,b)\rangle$ or $G' = (ab,C)$ or $G' = (b,C)$,

Suppose that $G' = (b,C)$, and set $H := \langle b,C\rangle$. Then $H' = G'$, and $\mathcal{C}_H(H') = C$, and $H/C = \langle bC\rangle$. Hence H satisfies the hypothesis of Lemma 2.2, i.e. $\mathbb{F}H$ is not Lie centre-by-metabelian, and certainly $\mathbb{F}G$ is neither. The same argument is valid for the case $G' = (ab,C)$.

Therefore we may assume that (a,b) has order 8, w.l.o.g. $(a,b) = x$. Then

$$\begin{aligned}\tau :=&[a + {}^b a, b + {}^a b] = [(1+x^{-1})a,(1+x)b]\\ =&(1+x^{-1})(1+x^{-1})ab + (1+x)(1+x^{-3})ba\\ =&\left((1+x^{-2}) + (1+x)(1+x^{-3})x^{-1}\right)ab = x^4\left(1+x+x^2+x^3\right)ab,\end{aligned}$$

and $[x,\tau] = x^4(1+x+x^2+x^3)[x,ab] = x^4(1+x+x^2+x^3)(1+x^4)xab = \langle x\rangle^+ ab \neq 0$. Hence $\mathbb{F}G$ is also not Lie centre-by-metabelian in this case. ∎

LEMMA 2.4: *Let the notation be as in 2.1, and assume that the image of φ is $\langle\alpha\rangle$. If $\mathbb{F}G$ is Lie centre-by-metabelian, then C is abelian.*

Proof: Suppose that $C' \neq 1$. Since $C' \subseteq \mathcal{Z}(G) \cap G' = \langle x^4 \rangle \cong Z_2$, we have $C' = \langle x^4 \rangle \subseteq \Phi(G')$.

Let $a \in G \setminus C$. Then $G/C = \langle aC \rangle$, ${}^a x = x^{-1}$, $G' = (a, C)$, and the map $(a, .)\colon C \to G'$ is an epimorphism. In particular, $U := (a,.)^{-1}(\langle x^2 \rangle) < C$. Let $d \in C \setminus \mathcal{Z}(C) \neq \emptyset$, then $V := \mathcal{C}_C(d) < C$. Therefore we may choose an element $c \in C \setminus (U \cup V) \neq \emptyset$. Then $(c,d) = x^4 = (d,c)$, and (a,c) has order 8, w.l.o.g. $(a,c) = x$. Hence

$$(\mathbb{F}G)'' \ni [a + {}^c a, c + {}^d c] = [\underbrace{(1+x^{-1})\,a}_{\in \mathcal{C}_{\mathbb{F}G}(c)}, \underbrace{(1+x^4)\,c}_{\in \mathcal{Z}(\mathbb{F}G)}]$$
$$= (1+x^{-1})(1+x^4)[a,c] = (1+x^{-1})(1+x^4)(1+x^{-1})ac = \langle x^2 \rangle^+ ac,$$

and $[c, \langle x^2 \rangle^+ ac] = \langle x^2 \rangle^+ [c,a]c = \langle x^2 \rangle^+ (1+x)cac = \langle x \rangle^+ cac \neq 0$. Therefore, $\mathbb{F}G$ is not Lie centre-by-metabelian. ∎

Remark 2.5: Let us now turn to the case $G' = \langle x, y \rangle \cong Z_2 \times Z_4$, where $x^2 = 1 = y^4$. Then $\mathrm{Aut}(G') = \langle \alpha, \beta \rangle \cong D_8$, where

$$\alpha\colon x \mapsto xy^2,\ y \mapsto xy, \qquad \beta\colon x \mapsto x,\ y \mapsto xy;$$

check that $\beta^2 = \mathrm{id}_{G'} = \alpha^4$, $\beta\alpha\beta^{-1} = \alpha^{-1}$. Set $C := \mathcal{C}_G(G')$ and let $\varphi\colon G/C \hookrightarrow \mathrm{Aut}(G')$ be the usual monomorphism. The algorithm in 1.3 leaves us with six (conjugacy classes of) "possible" images of φ. Since G/C is abelian, those are the (conjugacy classes of) the subgroups of order 2 or 4, namely $\langle \alpha^2, \beta \rangle$, $\langle \alpha \rangle$, $\langle \alpha^2, \alpha\beta \rangle$, $\langle \beta \rangle \sim \langle \alpha^2\beta \rangle$, $\langle \alpha^2 \rangle$, $\langle \alpha^3\beta \rangle \sim \langle \alpha\beta \rangle$ (here $\sim$ symbolizes conjugacy of subgroups).

We will show in Lemmata 2.6–2.11 that if $\mathbb{F}G$ is Lie centre-by-metabelian, then G/C is mapped onto $\langle \alpha^2 \rangle$, and that $C' \subseteq \Phi(G') = \langle y^2 \rangle$. (Note that $\langle \alpha^2 \rangle$ acts by element inversion on G', i.e. G then satisfies condition (iii) of Theorem 1.)

LEMMA 2.6: *Let the notation be as in 2.5, and assume that the image of φ is $\langle\alpha\rangle$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: Let $a \in G$ such that ${}^a h = \alpha(h)$ for all $h \in G'$, i.e. ${}^a x = xy^2$, ${}^a y = xy$. Then $G/C = \langle aC \rangle \cong Z_4$. Since $C' \subseteq \mathcal{Z}(G) \cap G' = \langle y^2 \rangle = \Phi(G')$, we have $G' = (a, C)C' = (a, C)$. Hence $(a, .)\colon C \to G'$ is an epimorphism. Choose an element $c \in C$ with $(a,c) = y$. Note that $(a,x) = y^2 = (x,a) \in \mathcal{Z}(G)$. Then

$$\begin{aligned}(\mathbb{F}G)'' \ni [c + {}^a c, a + {}^x a] &= [(1+y)c, (1+y^2)a] \\ &= (1+y^2)\big((1+y)ca + a(1+y)c\big) \\ &= (1+y^2)\big((1+y)ca + (1+xy)yca\big) \\ &= (1+y^2)(1+xy^2)ca = \langle x, y^2\rangle^+ ca,\end{aligned}$$

and $[c, \langle x,y^2\rangle^+ ca] = \langle x,y^2\rangle^+ c[c,a] = c\langle x,y^2\rangle^+(1+y)ca = c(G')^+ca \neq 0$. Therefore, $\mathbb{F}G$ is not Lie centre-by-metabelian. ∎

LEMMA 2.7: *Let the notation be as in 2.5, and assume that the image of φ is $\langle\beta\rangle$ or $\langle\alpha^2\beta\rangle$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: Since $\langle\beta\rangle$ and $\langle\alpha^2\beta\rangle$ are conjugate in $\mathrm{Aut}(G')$, we may (by renaming the elements of G' if necessary) w.l.o.g. assume that the image of φ is $\langle\beta\rangle$. Then there is an element $b \in G$ with ${}^b x = \beta(x) = x$ and ${}^b y = \beta(y) = xy$. Then $G/C = \langle bC\rangle \cong Z_2$ and $C' \subseteq \mathcal{Z}(G) \cap G' = \langle x, y^2\rangle = \Omega(G') \cong V_4$.

Now $G' = (b,C)C'$, so there is an element $c \in C$ such that $\tilde{y} := (b,c)$ has order 4, i.e. $\tilde{y} \in \{y, xy, y^{-1}, xy^{-1}\}$. In any case, ${}^b\tilde{y} = x\tilde{y}$. So we may w.l.o.g. assume that $\tilde{y} = y$. Then

$$\begin{aligned}(\mathbb{F}G)'' \ni [c + {}^b c, b + {}^y b] &= [(1+y)c, (1+x)b] = (1+x)\big((1+y)cb + b(1+y)c\big) \\ &= (1+x)\big((1+y)cb + (1+xy)ycb\big) = (1+x)(1+xy^2)cb = \langle x,y^2\rangle^+ cb,\end{aligned}$$

and $[c, \langle x,y^2\rangle^+ cb] = \langle x,y^2\rangle^+ c[c,b] = c\langle x,y^2\rangle^+(1+y)cb = c(G')^+cb \neq 0$. Therefore, $\mathbb{F}G$ is not Lie centre-by-metabelian. ∎

LEMMA 2.8: *Let the notation be as in 2.5, and assume that the image of φ is $\langle\alpha^3\beta\rangle$ or $\langle\alpha\beta\rangle$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: As in the preceding proof, we may w.l.o.g. assume that the image of φ is $\langle\alpha^3\beta\rangle$, and choose an element $b \in G$ with ${}^b x = (\alpha^3\beta)(x) = xy^2$ and ${}^b y = (\alpha^3\beta)(y) = y$. Then $G/C = \langle bC\rangle \cong Z_2$ and $C' \subseteq \mathcal{Z}(G) \cap G' = \langle y\rangle \cong Z_4$.

CASE 1: $C' = \langle y\rangle$. Here there are elements $c, \tilde{c} \in C$ such that $|\langle (c,\tilde{c})\rangle| = 4$; in particular, $(c,.)\colon C \to \langle y\rangle$ is an epimorphism. Then $U := (c,.)^{-1}(\langle y^2\rangle) < C$.

Now $G' = (b,C)C'$ implies that $(b,C) \not\subseteq \langle y\rangle$. Since the map $(b,.)\colon C \to G'$ is a homomorphism, this shows that $V := (b,.)^{-1}(\langle y\rangle) < C$.

Therefore we may choose an element $d \in C \setminus (U \cup V) \neq \emptyset$. Then $(c,d) \in \{y, y^{-1}\}$, and $(b,d) \in x\langle y\rangle$. By replacing d by its inverse if necessary, we may assume that $(c,d) = y$. Since for all $i \in \mathbb{Z}$, we have $(c^i b, d) = (c,d)^i (b,d) = y^i (b,d)$, we may replace b by $c^i b$ for a suitable i, and assume that $(b,d) = x$. Note that this does not change the action of b on G'. Then $(cb,d) = xy$, hence

$$(\mathbb{F}G)'' \ni [d + {}^{cb}d, c + {}^{d}c] = [(1+xy)d, (1+y^{-1})c] = (1+xy)(1+y^{-1})(dc+cd)$$
$$= (1+xy)(1+y^{-1})(1+y^{-1})cd = (1+xy)(1+y^2)cd = \langle xy\rangle^+ cd,$$

and $[c, \langle xy\rangle^+ cd] = \langle xy\rangle^+ c[c,d] = c\langle xy\rangle^+ (1+y)cd = c(G')^+ cd \neq 0$. Therefore, $\mathbb{F}G$ is not Lie centre-by-metabelian in this case.

CASE 2: $C' \subseteq \langle y^2 \rangle = \Phi(G')$. Then $G' = (b, C)$, i.e. the map $(b,.)\colon C \to G'$ is an epimorphism. If we choose elements $c, d \in C$ such that $(b,c) = x = (c,b)$, $(b,d) = y$, then

$$(\mathbb{F}G)'' \ni [c + {}^{b}c, b + {}^{c}b] = [(1+x)c, (1+x)b] = (1+x)\big((1+x)cb + b(1+x)c\big)$$
$$= (1+x)\big(1 + x + (1+xy^2)x\big)cb = (1+x)(1+y^2)cb = \langle x, y^2\rangle^+ cb.$$

Since $(cb,d) = (c,d)(b,d) \in y\langle y^2\rangle$, we then have

$$[d, \langle x,y^2\rangle^+ cb] = \langle x,y^2\rangle^+ [d, cb] = \langle x,y^2\rangle^+ (1 + (cb,d))dcb = (G')^+ dcb \neq 0.$$

Therefore, $\mathbb{F}G$ is also not Lie centre-by-metabelian in this case. ∎

LEMMA 2.9: *Let the notation be as in 2.5, and assume that the image of φ is $\langle \alpha^2, \beta\rangle$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: We may choose elements $a, b \in G$ with ${}^{a}h = \alpha^2(h) = h^{-1}$ for all $h \in G'$, and ${}^{b}x = \beta(x) = x$, ${}^{b}y = \beta(y) = xy$. Then $G/C = \langle aC, bC\rangle = \langle abC, bC\rangle \cong V_4$ and $C' \subseteq \mathcal{Z}(G) \cap G' = \langle x, y^2\rangle = \Omega(G') \cong V_4$. Moreover, $G' = \langle (a,b)\rangle (ab, C)(b,C)C' = \langle (a,b)\rangle (ab,C)(b,C)\Omega(G')$.

CASE 1: $\langle (a,b)\rangle \not\subseteq \Omega(G')$. With $\tilde{y} := (a,b)$ we have $G' = \langle \tilde{y}, x\rangle$, ${}^{a}\tilde{y} = \tilde{y}^{-1}$, ${}^{b}\tilde{y} = x\tilde{y}$, so by replacing y by $\tilde{y}$, we may assume that $(a,b) = y$. Then

$$(\mathbb{F}G)'' \ni [b + {}^{a}b, a + {}^{b}a] = [(1+y)b, (1+y^{-1})a]$$
$$= (1+y)(1+xy^{-1})ba + (1+y^{-1})(1+y^{-1})ab$$
$$= \big((1 + y + xy^{-1} + x) + (1+y^2)y\big)\, ba = (1+x)(1+y^{-1})ba =: \tau.$$

Since $(ba, y) = xy^2$, it follows that

$$[y, \tau] = (1+x)(1+y^{-1})[y, ba] = (1+x)(1+y^{-1})(1+xy^2)yba = (G')^+ ba \neq 0.$$

This shows that $\mathbb{F}G$ is not Lie centre-by-metabelian in this case.

CASE 2: $\langle (a,b)\rangle \subseteq \Omega(G')$. Then

$$(\beta, C) = (b, C) \nsubseteq \Omega(G') \quad \text{or} \quad (\alpha^2\beta, C) = (ab, C) \nsubseteq \Omega(G').$$

Since ${}^{\alpha}\beta = \alpha^2\beta$, we may conjugate both $\mathrm{Aut}(G')$ and G' by α if necessary, and assume w.l.o.g. that $(b, C) = (\beta, C) \nsubseteq \Omega(G')$.

Then there is an element $c \in C$ such that $\tilde{y} := (b, c) \in y\Omega(G')$. It is easy to see that this implies $G' = \langle x, \tilde{y}\rangle$, and $(b, b, c) = (b, \tilde{y}) = x$. So if we set $H := \langle b, C\rangle$, then $H' = G'$, $\mathcal{C}_H(H') = C$, and $H/C = \langle bC\rangle$. But then H satisfies the hypothesis of Lemma 2.7, so $\mathbb{F}H$ is not Lie centre-by-metabelian, and neither is $\mathbb{F}G$. ∎

LEMMA 2.10: *Let the notation be as in 2.5, and assume that the image of φ is $\langle \alpha^2, \alpha\beta\rangle = \langle \alpha^3\beta, \alpha\beta\rangle$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: We may choose elements $a, b \in G$ with ${}^{a}x = (\alpha^3\beta)(x) = xy^2$, ${}^{a}y = (\alpha^3\beta)(y) = y$, and ${}^{b}x = (\alpha\beta)(x) = xy^2$, ${}^{b}y = (\alpha\beta)(y) = y^3$; i.e. a acts trivially on $\langle y\rangle$ and by element inversion on $\langle xy\rangle$, and b acts by element inversion on $\langle y\rangle$ and trivially on $\langle xy\rangle$; note moreover that ${}^{ab}h = h^{-1}$ for all $h \in G'$.

Then $G/C = \langle aC, bC\rangle \cong V_4$, and $C' \subseteq \mathcal{Z}(G) \cap G' = \langle y^2\rangle = \Phi(G') \cong Z_2$, hence $G' = \langle (a,b)\rangle (a, C)(b, C)$.

CASE 1: $(a, C) = G'$. If we set $H := \langle a, C\rangle$, then $H' = G'$, and H/C is mapped onto $\langle \alpha^3\beta\rangle$ under φ. So H satisfies the hypotheses of Lemma 2.8, hence $\mathbb{F}H$ is not Lie centre-by-metabelian.

CASE 2: $(b, C) = G'$. Here $H := \langle b, C\rangle$ satisfies the hypotheses of Lemma 2.8, since H/C is mapped onto $\langle \alpha\beta\rangle$.

CASE 3: $(a, b) \notin \langle x, y^2\rangle = \Omega(G')$. Then ${}^{a}(a,b) = (a,b)^{-1}$ or ${}^{b}(a,b) = (a,b)^{-1}$. We only consider the case ${}^{b}(a,b) = (a,b)^{-1}$ here, since the case ${}^{a}(a,b) = (a,b)^{-1}$ can be handled completely analogously; we just have to switch y and xy, resp. a and b (this stems again from the fact that $\alpha^3\beta$ and $\alpha\beta$ are conjugate in $\mathrm{Aut}(G')$ under the automorphism β, which does switch y and xy). Then $(a, b) \in \{y, y^3\} \subseteq \mathcal{C}_G(a)$. By replacing a by $a^{-1} \in aC$ if necessary, we may assume that $(a, b) = y$. Then $(a, C)(b, C) \nsubseteq \langle y\rangle$, for otherwise $G' = \langle (a,b)\rangle (a, C)(b, C) \subseteq \langle y\rangle$.

If $(a, c) \notin \langle y\rangle$ for some $c \in C$, then

$$\begin{aligned}(\mathbb{F}G)'' \ni [b + {}^{a}b, ab + {}^{b}(ab)] &= [(1+y)b, (1+y^{-1})ab] \\ &= (1+y)(1+y)bab + (1+y^{-1})(1+y^{-1})abb \\ &= (1+y^2)(ba+ab)b = (1+y^2)(1+y^{-1})ab^2 = \langle y\rangle^{+}ab^2.\end{aligned}$$

Since $b^2 \in C$ we have $(b^2, c) \in C' \subseteq \langle y^2 \rangle$. Hence

$$[c, \langle y\rangle^+ ab^2] = \langle y\rangle^+(1 + (ab^2, c))cab^2 = \langle y\rangle^+(1 + (a, c))cab^2 = (G')^+ cab^2 \neq 0,$$

and $\mathbb{F}G$ is not Lie centre-by-metabelian.

So we may assume that $(a, C) \subseteq \langle y\rangle$. Then $z := (b, c) \notin \langle y\rangle$ for some $c \in C$, and ${}^a z = zy^2$. Furthermore,

$$\begin{aligned}(\mathbb{F}G)'' \ni [a + {}^b a, b + {}^c b] &= [(1 + y^{-1})a, (1 + z^{-1})b] \\ &= (1 + y^{-1})(1 + y^2 z^{-1})ab + (1 + z^{-1})(1 + y)ba \\ &= \left((1 + y^{-1})(1 + y^2 z^{-1})y + (1 + z^{-1})(1 + y)\right) ba \\ &= (1 + y)\left(1 + y^2 z^{-1} + 1 + z^{-1}\right) ba = \langle y\rangle^+ z^{-1} ba.\end{aligned}$$

Since $(ba, c) = {}^b(a, c)(b, c) \in \langle y\rangle z$, we have $[c, \langle y\rangle^+ z^{-1} ba] = \langle y\rangle^+ z^{-1}[c, ba] = z^{-1}\langle y\rangle^+(1+z)cba = (G')^+ cba \neq 0$. Therefore, $\mathbb{F}G$ is not Lie centre-by-metabelian in this case.

CASE 4: $(a, b) \notin \Phi(G')$. By case 3, $(a, b) \in \Omega(G') \setminus \Phi(G') = \langle x, y^2\rangle \setminus \langle y^2\rangle = \{x, xy^2\}$. By renaming x if necessary, we may even assume that $(a, b) = x$.

Then there exists an element $c \in C$ such that at least one of (a, c), (b, c) has order 4; by switching the roles of a and b, resp. y and xy as in case 3 if necessary, we may w.l.o.g. assume that $|\langle (a, c)\rangle| = 4$. Then $\langle (a, c)\rangle = \langle y\rangle$ or $\langle (a, c)\rangle = \langle xy\rangle$ (disappointingly there is no w.l.o.g.-ing anymore, since we might have switched y and xy already); by replacing c by c^{-1} if necessary we may assume that $(a, c) \in \{y, xy\}$.

Assume first that $(a, c) = y$. Note that $y \in \mathcal{Z}(\mathbb{F}[\langle a, c\rangle])$. Then

$$\begin{aligned}(\mathbb{F}G)'' \ni [ca + {}^c(ca), c + {}^a c] &= [(1 + y^{-1})ca, (1 + y)c] = (1 + y^{-1})(1 + y)[ca, c] \\ &= (1 + y^{-1})(1 + y)(1 + y)c^2 a = \langle y\rangle^+ c^2 a.\end{aligned}$$

Observe that $(b, c^2 a) = (b, c)^2 (b, a) \in \langle y^2\rangle x$, hence

$$[b, \langle y\rangle^+ c^2 a] = \langle y\rangle^+(1 + (b, c^2 a)) \cdot c^2 ab = \langle y\rangle^+ (1 + x)\, c^2 ab = (G')^+ c^2 ab \neq 0.$$

Hence $\mathbb{F}G$ is not Lie centre-by-metabelian.

Assume now that $(a, c) = xy$. Note that $(1 + x)ba = (1 + x)ab$. Then

$$\begin{aligned}(\mathbb{F}G)'' \ni [a + {}^c a, b + {}^a b] &= [(1 + xy^{-1})a, (1 + x)b] \\ &= (1 + xy^{-1})[a, (1 + x)b] \\ &= (1 + xy^{-1})\left((1 + xy^2)ab + (1 + x)ba\right) \\ &= (1 + xy^{-1})\left(1 + xy^2 + 1 + x\right) ab \\ &= (1 + xy^{-1})(1 + y^2)xab = \langle xy\rangle^+ xab.\end{aligned}$$

Now since $(b, xab) = (b,x)(b,a) \in \langle y^2 \rangle x$, we have

$$[b, \langle xy\rangle^+ xab] = \langle xy\rangle^+(1+(b,xab))xab^2 = \langle xy\rangle^+(1+x)xab^2 = (G')^+ xab^2 \neq 0.$$

Therefore $\mathbb{F}G$ is not Lie centre-by-metabelian, and case 4 is finished.

By the cases 1–4, we may assume that $(a, C) < G'$, $(b, C) < G'$, and $(a, b) \in \Phi(G')$. Then $G' = (a,C)(b,C)$, and consequently $(a,C) \cong Z_4$ or $(b,C) \cong Z_4$. W.l.o.g. $(a,C) \cong Z_4$, i.e. $(a,C) = \langle y\rangle$ (case 5 below) or $(a,C) = \langle xy\rangle$ (case 6).

Case 5: $(a,C) = \langle y\rangle$. Then $(a,.)\colon C \to \langle y\rangle$ is an epimorphism, and $U := (a,.)^{-1}(\langle y^2\rangle) < C$. The map $(b,.)\colon C \to G'$ is a homomorphism with image $(b,C) \not\subseteq \langle y\rangle$, so $V := (b,.)^{-1}(\langle y\rangle) < C$. Hence there is an element $c \in C \setminus (U \cup V) \neq \emptyset$. Then $(a,c) \in \{y, y^{-1}\}$ and $(b,c) \in x\langle y\rangle$. By replacing c by c^{-1} if necessary, we may even assume that $(a,c) = y$.

If (b,c) has order 4, then $(ba, c) = {}^b(a,c)(b,c) = y^{-1}(b,c) \in (b,c)\langle y\rangle = x\langle y\rangle$ has order 2. If we choose $\tilde{b} \in \{b, ba\}$ such that $(\tilde{b}, c)$ has order 2, and if we set $z := (\tilde{b}, c) = (c, \tilde{b})$, then $z \in x\langle y^2\rangle$ and $(1+z)\tilde{b}c = (1+z)c\tilde{b}$. Note that ${}^{\tilde{b}}y = y^{-1}$ and $(a,\tilde{b}) = (a,b) \in \langle y^2\rangle$ for any choice of $\tilde{b}$. Then

$$\begin{aligned}(\mathbb{F}G)'' \ni [c + {}^a c, \tilde{b} + {}^c\tilde{b}] &= [(1+y)c, (1+z)\tilde{b}] = (1+z)[(1+y)c, \tilde{b}] \\ &= (1+z)\left((1+y)c\tilde{b} + (1+y^{-1})\tilde{b}c\right) = (1+z)\left(1+y+1+y^{-1}\right)c\tilde{b} \\ &= (1+z)(1+y^2)yc\tilde{b} = \langle z, y^2\rangle^+ yc\tilde{b} = \langle x, y^2\rangle^+ yc\tilde{b}.\end{aligned}$$

Now $\langle x, y^2\rangle^+$ is central in $\mathbb{F}G$, since $\langle x, y^2\rangle \trianglelefteq G$. Moreover

$$(a, yc\tilde{b}) = (a,c)(a,b) \in y\langle y^2\rangle.$$

Hence

$$[a, \langle x,y^2\rangle^+ yc\tilde{b}] = \langle x,y^2\rangle^+[a, yc\tilde{b}] = \langle x,y^2\rangle^+(1+(a,yc\tilde{b}))yc\tilde{b}a = (G')^+ yc\tilde{b}a \neq 0.$$

Therefore $\mathbb{F}G$ is not Lie centre-by-metabelian in this case.

Case 6: $(a,C) = \langle xy\rangle$. Similarly as in case 5, we obtain an element $c \in C$ such that $(a,c) = xy$ and $z := (b,c) \notin \langle xy\rangle$. Then

$$\begin{aligned}(\mathbb{F}G)'' \ni [ca + {}^a(ca), a + {}^c a] &= [(1+xy)ca, (1+xy^{-1})a] \\ &= (1+xy)(1+xy)caa + (1+xy^{-1})(1+xy^{-1})aca \\ &= (1+y^2)(ca+ac)a = (1+y^2)(1+xy)ca^2 = \langle xy\rangle^+ ca^2.\end{aligned}$$

Now $(b, ca^2) = (b,c)(b,a^2) = (b,c)(b,a)^2 = (b,c) = z$, and so $[b, \langle xy\rangle^+ ca^2] = \langle xy\rangle^+(1+(b,ca^2))ca^2 b = \langle xy\rangle^+(1+z)ca^2b = (G')^+ca^2b \neq 0$. So $\mathbb{F}G$ is also not Lie centre-by-metabelian in this last case. ∎

LEMMA 2.11: *Let the notation be as in 2.5, and assume that the image of φ is $\langle\alpha^2\rangle$. If $C' \not\subseteq \Phi(G')$, then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: We may choose an element $a \in G$ with ${}^a h = \alpha^2(h) = h^{-1}$ for all $h \in G'$. Then $G/C = \langle aC\rangle \cong Z_2$ and $C' \subseteq \mathcal{Z}(G) \cap G' = \langle x, y^2\rangle \cong V_4$.

Since $C' \not\subseteq \Phi(G') = \langle y^2\rangle$, there are elements $c, \tilde{c} \in C$ such that $|\langle(c,\tilde{c})\rangle| \notin \langle y^2\rangle$. For the homomorphism $(c,.)\colon C \to \langle x, y^2\rangle$, this implies that $U := (c,.)^{-1}(\langle y^2\rangle) < C$.

Now $G' = (a,C)C'$ implies that $(a,C) \not\subseteq \langle x, y^2\rangle$. Since the map $(a,.)\colon C \to G'$ is a homomorphism, this shows that $V := (a,.)^{-1}(\langle x, y^2\rangle) < C$.

Therefore we may choose an element $d \in C \setminus (U \cup V) \neq \emptyset$. Then $(c,d) \in x\langle y^2\rangle$, and $(a,d) \in y\langle x, y^2\rangle$; w.l.o.g. $(c,d) = x$, $(a,d) = y$ (rename x and y if necessary). We compute

$$\begin{aligned}(\mathbb{F}G)'' \ni [d + {}^c d, a + {}^d a] &= [(1+x)d, (1+y^{-1})a] = (1+x)(1+y^{-1})(da+ad)\\ &= (1+x)(1+y^{-1})(1+y^{-1})ad = (1+x)(1+y^2)ad = \langle x, y^2\rangle^+ ad,\end{aligned}$$

and $[d, \langle x, y^2\rangle^+ ad] = \langle x, y^2\rangle^+ [d,a]d = \langle x, y^2\rangle^+ (1+y)dad = (G')^+ dad \neq 0$. This shows that $\mathbb{F}G$ is not Lie centre-by-metabelian. ∎

3. Commutator subgroups of order 16

We now want to establish Theorem 1 for all groups G with $|G'| \mid 16$. By the preceding two sections, it suffices to suppose that $|G'| = 16$, and study the $1+5+3=9$ cases mentioned in the lower half of the table in 1.3. In particular, we may assume that G' is isomorphic to $Z_4 \times Z_4$, or $Z_2 \times Z_8$, or $V_4 \times Z_4$.

Remark 3.1: Suppose that G is a counterexample to Theorem 1 with $G' \cong Z_4 \times Z_4$, and set $C := \mathcal{C}_G(G')$. Then by 1.3, we only have to study one particular action of G on G'; this turns out to be element inversion. So $|G:C| = 2$, and if we fix an element $a \in G \setminus C$, then ${}^a h = h^{-1}$ for all $h \in G'$.

Moreover, $C' \subseteq G' \cap \mathcal{Z}(G) = \Phi(G')$, and thus $G' = (a,C)C' = (a,C)$. Consequently, $(a,.)\colon C \to G'$ is an epimorphism.

If C is abelian, then G satisfies condition (iv) of Theorem 1, so it is not a counterexample. Therefore $\mathcal{Z}(C) < C$. Certainly $U := (a,.)^{-1}(\Phi(G')) < C$, so we may choose an element $c \in C \setminus (U \cup \mathcal{Z}(C)) \neq \emptyset$.

We set $x := (a,c) \notin \Phi(G')$. Then $V := (a,.)^{-1}(\langle x, \Phi(G')\rangle) < C$ and $W := \mathcal{C}_C(c) < C$. Choose an element $d \in C \setminus (V \cup W) \neq \emptyset$.

If we set $y := (a,d)$, then $G' = \langle x, y\rangle$, and $(c,d) \in \langle x^2, y^2\rangle \setminus \{1\}$. Then

$$\begin{aligned}(\mathbb{F}G)'' \ni [d + {}^a d, ca + {}^a(ca)] &= [(1+y)d, (1+x)ca] \\ &= (1+x)\left((1+y)dca + (1+y^{-1})(ca,d)dca\right) \\ &= (1+x)\left((1+y) + (1+y^{-1})(c,d)y\right) dca \\ &= (1+x)(1+y)(1+(c,d))dca =: \tau.\end{aligned}$$

If $(c,d) = x^2$, then $\tau = dc(1+y)\langle x\rangle^+ a$, and hence $[y,\tau] = dc(1+y)\langle x\rangle^+[y,a] = dc(1+y)\langle x\rangle^+(1+y^2)ya = dc(G')^+a \neq 0$, contradiction.

Consequently $\tilde{y}^2 := (c,d) \in \{y^2, x^2y^2\}$. Then $\tau = dc(1+x)(1+y)(1+\tilde{y}^2)a$, and $[x,\tau] = dc(1+x)(1+y)(1+\tilde{y}^2)[x,a] = dc(1+x)(1+y)(1+\tilde{y}^2)(1+x^2)xa = dc(G')^+a \neq 0$, contradiction.

Remark 3.2: Suppose now that G is a counterexample to Theorem 1 with $G' = \langle x,y\rangle \cong Z_2 \times Z_8$, where $x^8 = 1 = y^2$. Then $\mathbb{F}G$ is Lie centre-by-metabelian, and all subgroups of index 2 in G are nonabelian.

Set $C := \mathcal{C}_G(G')$, and map G/C to $\mathrm{Aut}(G') \cong Z_2 \times D_8$ in the usual way. The reductions of the algorithm described in 1.3 give the following possible images of G/C in $\mathrm{Aut}(G')$:

$$\begin{array}{llll} & \langle\alpha,\beta,\gamma\rangle & & \cong Z_2\times Z_2\times Z_2, \\ \langle\alpha\beta,\alpha\gamma\rangle \sim \langle\alpha\beta,\gamma\rangle\,, & \langle\alpha\gamma,\alpha\beta\gamma\rangle\,, & \langle\alpha,\gamma\rangle\sim\langle\alpha,\beta\gamma\rangle & \cong Z_2\times Z_2, \\ & \langle\alpha\gamma\rangle\sim\langle\alpha\beta\gamma\rangle & & \cong Z_2,\end{array}$$

where $\alpha\colon x\mapsto x^3,\ y\mapsto y$, $\beta\colon x\mapsto x^5,\ y\mapsto y$, $\gamma\colon x\mapsto x,\ y\mapsto yx^4$.

Note that $(\alpha\gamma, G') = (\langle\alpha,\beta,\gamma\rangle, G') = \langle x^2\rangle = \Phi(G')$. So in any case we have $(G,G') = \Phi(G')$, i.e. $G/\Phi(G')$ has class 2. So for all elements $g,h\in G$, we obtain $\langle g,h\rangle'\,\Phi(G')/\Phi(G') = \langle (g,h)\rangle\,\Phi(G')/\Phi(G')$. Hence $\langle g,h\rangle' \subseteq \langle (g,h)\rangle\,\Phi(G')$; in particular:

$$(*).\qquad \forall g,h\in G\colon\ |\langle (g,h)\rangle| = 8 \Longrightarrow \langle g,h\rangle' = \langle (g,h)\rangle$$

Assumption: $|G:C|\geq 4$.

Let $a,b\in G$ such that $|\langle (a,b)\rangle| = 8$. By renaming x and y if necessary, we may assume that $(a,b) = x$ (note however that this "fixes" the image of G/C in $\mathrm{Aut}(G')$ in the sense that we may not replace it by a conjugate subgroup).

Set $H := \langle a,b\rangle$, then $H' = \langle x\rangle$ by $(*)$. The results of section 2 together with lemma 0.2 imply that H acts by element inversion on H', i.e. $|H:\mathcal{C}_H(x)| = 2$, and that $\mathcal{C}_H(x)$ is abelian. W.l.o.g. $H/\mathcal{C}_H(x) = \langle a\mathcal{C}_H(x)\rangle$. Then ${}^b x = x$ or ${}^b x = {}^a x$. In the latter case, $ba\in\mathcal{C}_H(x)$. Since $(a,b) = (a,ba)$, we may replace b by ba, and thus assume that $b\in\mathcal{C}_H(x)$.

Now $|G:C| \geq 4$ implies that there is an element $g \in G$ such that $4 = |\langle aC, gC\rangle|$. If $bC \in \langle aC\rangle$, then $\langle aC, gC\rangle = \langle aC, gbC\rangle$, and either (a,g) or $(a,bg) = (a,b)\,{}^b(a,g) = x(a,g)$ has order 8, w.l.o.g. $|\langle(a,g)\rangle| = 8$. So after replacing b by g if necessary and working through the preceding paragraphs again, we may assume that $|\langle aC, bC\rangle| = 4$, $(a,b) = x$, ${}^a x = x^{-1}$, and ${}^b x = x$. Hence $\varphi(aC) \in \{\alpha\beta, \alpha\beta\gamma\}$, and $\varphi(bC) = \gamma$.

Set $K := \langle a, b, y\rangle$. Note that $(G, y) = \langle x^4\rangle$, so $K' = H' = \langle x\rangle$. As above, $\mathcal{C}_K(x)$ must be abelian. But $b, y \in \mathcal{C}_K(x)$, and $(b,y) = (\gamma, y) = x^4 \neq 1$, contradiction.

This shows that $|G:C| = 2$.

Then G/C is mapped onto $\langle\alpha\gamma\rangle$ or $\langle\alpha\beta\gamma\rangle$. Now $\mathcal{C}_{G'}(\alpha\gamma) = \mathcal{C}_{G'}(\alpha\beta\gamma) = \langle yx^2\rangle$, i.e. $C' \subseteq \mathcal{Z}(G) \cap G' = \langle yx^2\rangle \cong Z_4$. If $C' = 1$, then C is an abelian subgroup of index 2 in G, and G is not a counterexample. If $|C'| = 2$, then $\mathbb{F}G$ is not Lie centre-by-metabelian by lemma 1.2, so G is also not a counterexample. Hence $C' = \langle yx^2\rangle$.

Set $N := \langle x^4\rangle = \Phi(\Phi(G')) \trianglelefteq G$, and $\bar{H} := HN/N$ for all $H \leq G$, and $\bar{g} := gN$ for all $g \in G$. Then $\bar{G}' \cong Z_2 \times Z_4$, and $\langle \bar{x}\bar{y}^2\rangle = \bar{C}' \subseteq \mathcal{C}_{\bar{G}}(\bar{G}')'$; in particular, $\mathcal{C}_{\bar{G}}(\bar{G}')' \not\subseteq \langle\bar{x}^2\rangle = \Phi(\bar{G}')$. But then $\bar{G}$ is a counterexample to Theorem 1 with $|\bar{G}'| = 8$, in contradiction to the results of section 2.

Remark 3.3: Let G be a group with $G' = \langle x, y, z\rangle \cong V_4 \times Z_4$, where $x^2 = y^2 = z^4 = 1$. As usual, set $C := \mathcal{C}_G(G')$ and map G/C to $\mathrm{Aut}(G')$. By 1.3, G may only be a counterexample to Theorem 1, if the image of G/C is (conjugate to) one of the following elementary abelian 2-groups:

$$\langle\alpha\rangle, \quad \langle\beta,\gamma\rangle, \quad \langle\alpha,\beta,\gamma\rangle,$$

where α: $z \mapsto z^3$, x, y fixed, β: $x \mapsto xz^2$, y, z fixed, γ: $y \mapsto yz^2$, x, z fixed. (In fact, $\langle\alpha\rangle$ and $\langle\alpha,\beta,\gamma\rangle$ are normal in $\mathrm{Aut}(G')$, while $\langle\beta,\gamma\rangle$ is not.)

We will study those cases in 3 separate lemmata.

Lemma 3.4: *Given the notation of 3.3, suppose that $|G:C| = 2$, and that $\mathbb{F}G$ is Lie centre-by-metabelian. Then G contains an abelian subgroup of index 2.*

Proof: By 3.3, G/C is mapped onto $\langle\alpha\rangle$.

Then $C' \subseteq \mathcal{Z}(G) \cap G' = \langle x, y, z^2\rangle$. Set $N := \langle x\rangle \trianglelefteq G$. Then $G'/N \cong Z_2 \times Z_4$. Since $\mathbb{F}[G/N]$ is Lie centre-by-metabelian, section 2 implies that $C'/N \subseteq \mathcal{C}_{G/N}(G'/N)' \subseteq \Phi(G'/N) = \langle z^2N\rangle$. Therefore $C' \subseteq \langle x, z^2\rangle$. Similarly $C' \subseteq \langle y, z^2\rangle$, so together we have $C' \subseteq \langle x, z^2\rangle \cap \langle y, z^2\rangle = \langle z^2\rangle \cong Z_2$.

Now Lemma 1.2 implies that $|C'| \neq 2$. Hence $C' = 1$, and C is abelian. ∎

LEMMA 3.5: *Given the notation of 3.3, suppose that* $|G : C| = 4$. *Then* $\mathbb{F}G$ *is not Lie centre-by-metabelian.*

Proof: By 3.3, we may assume that G/C is mapped onto $\langle\beta,\gamma\rangle$. Then $C' \subseteq \mathcal{Z}(G) \cap G' = \langle z\rangle$. We write $G/C = \langle aC, bC\rangle$ where ${}^a h = \gamma(a)$, ${}^b h = \beta(h)$ for all $h \in G'$. Then ${}^a x = x$, ${}^a y = yz^2$, ${}^a z = z$, and ${}^b x = xz^2$, ${}^b y = y$, ${}^b z = z$.

Set $H := \langle b, C\rangle$. Then $H' = (b,C)C'$ and $C \subseteq \mathcal{C}_H(H')$.

CASE 1: $H' = G'$. Then H does not act by element inversion on H', hence C cannot be abelian by Lemma 0.2, so by Lemma 3.4, $\mathbb{F}H$ is not Lie centre-by-metabelian.

CASE 2: $H' \cong Z_2 \times Z_2 \times Z_2$. Then $H' = \langle x, y, z^2\rangle$, hence $b \in H \setminus \mathcal{C}_H(H')$, and so $\mathrm{cl}(H) > 2$. By section 2, $\mathbb{F}H$ is not Lie centre-by-metabelian.

CASE 3: $H' \cong Z_2 \times Z_4$.

If H does not act by element inversion on H', or if $C' \not\subseteq \Phi(H')$, then $\mathbb{F}H$ is not Lie centre-by-metabelian by Lemma 0.2 and section 2.

So we may assume that ${}^b h = h^{-1}$ for all $h \in H'$, and that $C' \subseteq \langle z^2\rangle$. It is easy to check that $\{h \in G' \colon {}^b h = h^{-1}\} = \langle y, xz\rangle$. Therefore $\langle y, xz\rangle = H' = (b,C)C' = (b,C)$, i.e. $(b,.)\colon C \to H'$ is an epimorphism.

Now if $(a,b) \notin H'$, then $(a,b) \in xH'$. Note that $(ca,b) = (c,b)(a,b)$ for all $c \in C$, so by replacing a by a suitable element of Ca if necessary, we may assume that $(a,b) = x = (b,a)$. Let $c \in C$ such that $(b,c) = y = (c,b)$. Then

$$\begin{aligned}(\mathbb{F}G)'' \ni [a + {}^b a, b + {}^c b] &= [(1+x)a, (1+y)b] \\ &= (1+x)(1+yz^2)ab + (1+y)(1+xz^2)ba \\ &= (1+x)(1+yz^2)xba + (1+y)(1+xz^2)ba \\ &= \left(x + yz^2 + y + xz^2\right) ba \\ &= (x+y)(1+z^2)ba = x\langle xy, z^2\rangle^+ ba,\end{aligned}$$

and

$$\begin{aligned}[a, x\langle xy, z^2\rangle^+ ba] = x\langle xy, z^2\rangle^+ [a,b]a &= x\langle xy, z^2\rangle^+ (1+x)ba^2 \\ &= \langle x, y, z^2\rangle^+ ba^2 \neq 0,\end{aligned}$$

i.e. $\mathbb{F}H$ is not Lie centre-by-metabelian.

Hence we may assume that $(a,b) \in H' = (b,C)$. As above, we may replace a by a suitable element of Ca and assume that $(a,b) = 1$.

Then $G' = \langle (a,b)\rangle\,(a,C)(b,C)C' = (a,C)H'$. Since $G' \not\subseteq H'$, we have $(a,C) \not\subseteq H'$. Then $(a,.)\colon C \to G'$ is a homomorphism with $U := (a,.)^{-1}(H') < C$.

Similarly, $(b, .)\colon C \to H'$ is an epimorphism, i.e. $V := (b,.)^{-1}(\langle y, z^2\rangle) < C$. Let $c \in C \smallsetminus (U \cup V) \neq \emptyset$, then $(b,c) \in H'$ has order 4, and $(a,c) \notin H'$.

Then $z^2 \in \langle (a,c),\, (b,c)\rangle \cong Z_2 \times Z_4$. Since $G/\langle z^2\rangle$ has class 2, also $\langle a,b,c\rangle / \langle z^2\rangle$ has class 2. Then

$$\langle a,b,c\rangle' / \langle z^2\rangle = \langle (a,b),\, (a,c),\, (b,c)\rangle / \langle z^2\rangle = \langle (a,c),\, (b,c)\rangle / \langle z^2\rangle .$$

Therefore $\langle a,b,c\rangle' = \langle (a,c),\, (b,c)\rangle$, in particular $\langle a,b,c\rangle' \cong Z_2 \times Z_4$. Now b does neither act trivially on $\langle a,b,c\rangle'$, since ${}^{b}(b,c) = (b,c)^{-1}$, nor by element inversion, since $(a,c) \notin H' = \{h \in G'\colon {}^{b}h = h^{-1}\}$. By section 2, $\mathbb{F}\langle a,b,c\rangle$ is not Lie centre-by-metabelian.

By the cases 1–3, we may assume that $|H'| \le 4$. Furthermore $z^2 = (b,x) \in (b,C) \subseteq (b,C)C' = H'$.

For $K := \langle a, C\rangle$, we argue similarly as in the cases above to show that $|K'| \ge 8$ implies that $\mathbb{F}G$ is not Lie centre-by-metabelian. So we may assume that $|K'| \le 4$. Note that $z^2 = (a,y) \in (a,C)C' = K'$.

Then $|H'K'| \le 8$. Now $G' = \langle (a,b)\rangle\, (a,C)(b,C)C' = \langle (a,b)\rangle\, H'K'$. For order reasons, $Z_2 \times Z_2 \times Z_2 \cong G'/\langle z^2\rangle = \langle (a,b), z^2\rangle / \langle z^2\rangle \times H'/\langle z^2\rangle \times K'/\langle z^2\rangle$. Hence $C' \subseteq H' \cap K' = \langle z^2\rangle$. Then $H' = (b,C)$, $K' = (a,C)$, and $|K'| = |H'| = 4$.

Suppose that $(b,C) \cong Z_4$. As usual, we find an element $c \in C$ such that $w := (b,c)$ has order 4, and $(a,c) \notin \langle w\rangle$. Note that $1 = (b^{-1}b, c) = {}^{b^{-1}}(b,c)(b^{-1},c) = {}^{b}w(b^{-1},c)$. Hence $(b^{-1},c) \in \{w^{-1},\, {}^{b}w^{-1}\} \subseteq \{w^{\pm 1}\}$. Since $(b^{-1}, c^{-1}) = (b^{-1},c)^{-1}$, there is an element $d \in \{c, c^{-1}\}$ such that $(b^{-1}, d) = w$. Then also $(cb^{-1}, d) = w$, and thus

$$\begin{aligned}(\mathbb{F}G)'' \ni [b + {}^{c^{-1}}b,\ cb^{-1} + {}^{d^{-1}}(cb^{-1})] &= [(1+w)b, (1+w)cb^{-1}] \\ &= (1+w)(1+{}^{b}w)bcb^{-1} + (1+w)(1+{}^{b}w)c = (1+w)(1+{}^{b}w)(c + {}^{b}c) \\ &= (1+w)(1+{}^{b}w)(1+w)c = (1+w^2)(1+w^{\pm 1}) = \langle w\rangle^{+}c,\end{aligned}$$

and $[a, \langle w\rangle^{+}c] = \langle w\rangle^{+}(1 + (a,c))ca \neq 0$, so $\mathbb{F}G$ is not Lie centre-by-metabelian.

So we may assume that $H' = (b,C) \cong V_4$, and similarly $K' = (a,C) \cong V_4$. Then $(a,b) \notin \langle x,y,z^2\rangle = H'K'$. As usual, we find elements $c,d \in C$ such that $(b,d) = z^2$, and $t := (b,c) \in H' \smallsetminus \langle z^2\rangle$, and $s := (a,c) \in K' \smallsetminus \langle z^2\rangle$. Then $\langle s,t,z^2\rangle = H'K' = \langle x,y,z^2\rangle$. Note that b commutes with s modulo $\langle z^2\rangle \subseteq \mathcal{Z}(G)$, so

$$\begin{aligned}(\mathbb{F}G)'' \ni [b + {}^{d}b, c + {}^{a}c] &= [(1+z^2)b, (1+s)c] = (1+z^2)(1+s)[b,c] \\ &= (1+z^2)(1+s)(1+t)cb = \langle z^2, s, t\rangle^{+}cb = \langle x,y,z^2\rangle^{+}cb.\end{aligned}$$

Since a and c commute modulo $\langle x, y, z^2\rangle$, we furthermore have

$$\begin{aligned}[a, \langle x, y, z^2\rangle^+ cb] &= \langle x, y, z^2\rangle^+ [a, cb] = c\langle x, y, z^2\rangle^+ [a, b] \\ &= c\langle x, y, z^2\rangle^+ (1 + (a, b))ba = c(G')^+ ba \neq 0.\end{aligned}$$

Hence $\mathbb{F}G$ is not Lie centre-by-metabelian. ∎

LEMMA 3.6: *Given the notation of 3.3, suppose that $|G : C| = 8$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: By 3.3, G/C is mapped onto $\langle \alpha, \beta, \gamma\rangle = \langle \beta, \gamma, \alpha\beta\gamma\rangle$. Choose elements $g, h, k \in G$ such that ${}^g v = \beta(v)$, ${}^h v = \gamma(v)$, ${}^k v = \alpha\beta\gamma(v)$ for all $v \in G'$. Then

$$\begin{array}{lll} {}^g x = xz^2, & {}^g y = y, & {}^g z = z, \\ {}^h x = x, & {}^h y = yz^2, & {}^h z = z, \\ {}^k x = xz^2, & {}^k y = yz^2, & {}^k z = z^3. \end{array}$$

It is easy to check that

$$(*) \qquad \begin{array}{ll} C_{G'}(g) = \langle y, z\rangle, & \{v \in G' \colon {}^g v = v^{-1}\} = \langle y, xz\rangle, \\ C_{G'}(h) = \langle x, z\rangle, & \{v \in G' \colon {}^h v = v^{-1}\} = \langle x, yz\rangle, \\ C_{G'}(k) = \langle xy, xz\rangle, & \{v \in G' \colon {}^k v = v^{-1}\} = \langle xy, z\rangle. \end{array}$$

Set $H := \langle g, h, C\rangle$. Then $H' = \langle (g, h)\rangle (g, C)(h, C)C'$. If $H' = G'$, then $\mathbb{F}H$ is not Lie centre-by-metabelian by Lemma 3.5. Hence we may assume that $|H'| \leq 8$. Then H is not a counterexample to Theorem 1. If $H' \cong Z_2 \times Z_2 \times Z_2$, then $H' = \langle x, y, z^2\rangle$. But since H does not act trivially on $\langle x, y, z^2\rangle$, the group algebra $\mathbb{F}H$ is not Lie centre-by-metabelian. Suppose next that $H' \cong Z_2 \times Z_4$. If $\mathbb{F}H$ was Lie centre-by-metabelian, then by section 2, $\langle g, h\rangle$ would have to act by element inversion on H', which is impossible by (*). Hence we may assume that $|H'| \leq 4$.

If we set $K := \langle g, k, C\rangle$ and $L := \langle h, k, C\rangle$, we similarly may assume that

$$\begin{aligned} K' &= \langle (g, k)\rangle (g, C)(k, C)C' \text{ and } |K'| \leq 4, \\ L' &= \langle (h, k)\rangle (h, C)(k, C)C' \text{ and } |L'| \leq 4. \end{aligned}$$

Note that $z^2 \in (g, G') \cap (h, G') \cap (k, G') \subseteq (g, C) \cap (h, C) \cap (k, C) \subseteq H' \cap L' \cap K'$. Moreover, since $G' = \langle (g, h), (g, k), (h, k)\rangle (g, C)(h, C)(k, C)C'$, we have $G' = H'K'L'$. For order reasons,

$$Z_2 \times Z_2 \times Z_2 \cong G'/\langle z^2\rangle = H'/\langle z^2\rangle \times K'/\langle z^2\rangle \times L'/\langle z^2\rangle.$$

In particular,

$$(g,C)C' \subseteq H' \cap K' = \langle z^2 \rangle = \Phi(G'),$$
$$(h,C)C' \subseteq H' \cap L' = \langle z^2 \rangle,$$
$$(k,C)C' \subseteq K' \cap L' = \langle z^2 \rangle.$$

Therefore $G' = \langle (g,h),(g,k),(h,k)\rangle$. Choose $r,s \in \{g,h,k\}$ such that $w := (r,s)$ has order 4, and let $t \in \{g,h,k\} \setminus \{r,s\}$. Then $w^2 = z^2$, and

$$(**) \qquad G' = \langle (r,s),(r,t),(s,t)\rangle\,,\ (r,s) = w \text{ with } |\langle w \rangle| = 4.$$

Note that (r,w), $(s,w) \in (G,G') = \langle w^2 \rangle$.

Assume first that $(r,w) = 1 = (s,w)$, then

$$(\mathbb{F}G)'' \ni [r + {}^{s}r, s + {}^{r}s] = [(1+w^3)r,(1+w)s] = (1+w^3)(1+w)[r,s]$$
$$= (1+w^3)(1+w)(1+w)sr = \langle w \rangle^+ sr.$$

Now $\langle w \rangle \trianglelefteq G$, i.e. $\langle w \rangle^+ \in \mathcal{Z}(\mathbb{F}G)$. Moreover $(sr,t) \notin \langle w \rangle$. Hence $[t, \langle w \rangle^+ sr] = \langle w \rangle^+[t,sr] = \langle w \rangle^+(1+(t,sr))srt \neq 0$. Therefore, $\mathbb{F}G$ is not Lie centre-by-metabelian in this case.

So we may w.l.o.g. assume that $(r,w) = w^2 = z^2$. By possibly replacing s by sr, we even may assume that $(s,w) = w^2$ (since $(r,sr) = (r,s)$ and $(sr,t) = {}^{s}(r,t)(s,t) \in \langle w^2,(r,t)\rangle (s,t) \subseteq \langle (r,s),(r,t)\rangle (s,t)$, this does not change (**)). Then

$$(\mathbb{F}G)'' \ni [r + {}^{s}r, s + {}^{r}s] = [(1+w^3)r,(1+w)s]$$
$$= (1+w^3)(1+w^3)rs + (1+w)(1+w)sr$$
$$= (1+w^2)(rs+sr) = (1+w^2)(1+w)sr = \langle w \rangle^+ sr.$$

As before, $[t,\langle w \rangle^+ sr] = \langle w \rangle^+(1+(t,sr))srt \neq 0$. Hence $\mathbb{F}G$ is also not Lie centre-by-metabelian in this case. ∎

4. Finish

Let us collect all instances in which we already know that Theorem 1 holds: In [9], we verified it for nilpotent groups of class at most 2, and for groups G with $\exp(G') \leq 2$. The preceding sections 2 and 3 then established it for groups G with $|G'| \mid 16$. We are now prepared to finally prove it for arbitrary groups.

LEMMA 4.1: *Let $\mathbb{F}G$ be a Lie centre-by-metabelian group algebra. If G' is a finite 2-group of order at least 16, then G has an abelian subgroup of index 2.*

Proof: We argue by induction on $|G'|$. By the results of section 3, we may assume that $|G'| \geq 32$. By [9], we may also assume that $\exp(G') \neq 2$. Hence $\Phi(G') \neq 1$.

It now suffices to show that $G' \cap \mathcal{Z}(G) \neq 1$, because then we may factor out an involution in $G' \cap \mathcal{Z}(G)$, and obtain the result from Lemma 1.2.

Set $C := \mathcal{C}_G(G')$. Note that G/C is finite, since G' is finite.

If G/C is a 2-group, then $G' \rtimes G/C$ is also a (finite) 2-group, and $(G' \rtimes 1) \cap \mathcal{Z}(G' \rtimes G/C) \neq 1$, i.e. $G' \cap \mathcal{Z}(G) \neq 1$.

So suppose that G/C is not a 2-group. Then by 1.1, G/C has a Hall $2'$-subgroup which centralizes $\Phi(G')$. Therefore $G/\mathcal{C}_G(\Phi(G'))$ is a finite 2-group, and similarly as above, we find that $1 \neq \Phi(G') \cap \mathcal{Z}(G) \subseteq G' \cap \mathcal{Z}(G)$. ∎

Remark 4.2: The preceding lemma shows that if there is a counterexample G to theorem 1, then G' is not a finite 2-group. Then by [7], G contains a subgroup A of index 2 such that A' is a finite 2-group. We set

$$\mathfrak{A}(G) := \{A \leq G\colon |G:A| = 2 \text{ and } A' \text{ is a finite 2-group}\} \neq \emptyset,$$
$$\mathfrak{a}(G) := \min\{|A'| : A \in \mathfrak{A}(G)\} \in \mathbb{N},$$

and we have to show that if $\mathbb{F}G$ is Lie centre-by-metabelian, then $\mathfrak{a}(G) = 1$. But before we do so in 4.4, let us quickly insert another lemma:

LEMMA 4.3: *Let G be a group with a normal subgroup U that is isomorphic to V_4. Set $C := \mathcal{C}_G(U)$, and let $\varphi\colon G/C \to \operatorname{Aut}(U) \cong S_3$ be the usual monomorphism. If φ is surjective, then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: We write $\langle x, y\rangle = U \cong V_4$. If φ is surjective, then there are elements $g, h \in G$ with ${}^g x = y$, ${}^g y = x$, ${}^h x = y$, ${}^h y = xy$. Set $a := (g,h)$, then ${}^a x = y$ and ${}^a y = xy$; in particular, $a \in G \setminus U$. Then

$$\begin{aligned}\rho :=&[x + {}^a x, h + {}^g h] = [x+y, h + (g,h)h] = (x+y)(h+ah) + (h+ah)(x+y)\\ =&(x+y)h + (x+y)ah + (y+xy)h + (xy+x)ah = (x+ya+xy+xya)h,\end{aligned}$$

and

$$\begin{aligned}[x,\rho] =&[x,(x+xy)h] + [x,(y+xy)ah] = (x+xy)[x,h] + (y+xy)[x,ah]\\ =&(x+xy)(1+(h,x))xh + (y+xy)(1+(ah,x))xah\\ =&\left((x+xy)(1+xy)x + (y+xy)(1+y)xa\right)h = \left(U^+ + U^+a\right)h \neq 0.\end{aligned}$$ ∎

LEMMA 4.4: *Let G be a group. Suppose that G' is not a finite 2-group, and that $\mathfrak{a}(G) \geq 2$. Then $\mathbb{F}G$ is not Lie centre-by-metabelian.*

Proof: We argue by induction on $\mathfrak{a}(G)$, which clearly is a power of 2. If $\mathfrak{a}(G) = 2$, then $\mathbb{F}G$ is not Lie centre-by-metabelian by 1.2.

So we may assume that $\mathfrak{a}(G) \geq 4$. Let $A \in \mathfrak{A}(G)$ such that $|A'| = \mathfrak{a}(G)$. Note that A itself is not a counterexample to Theorem 1 by 4.1.

Assume there is a normal subgroup N of G with $1 < N < A'$. Then also $(G/N)' = G'/N$ is not a finite 2-group. Let $B/N \in \mathfrak{A}(G/N)$ with $|(B/N)'| = \mathfrak{a}(G/N)$. Since then $B \in \mathfrak{A}(G)$ and $A/N \in \mathfrak{A}(G/N)$, we have $\mathfrak{a}(G)/|N| \leq |B'N|/|N| = |(B/N)'| = \mathfrak{a}(G/N) \leq |A' : N| = \mathfrak{a}(G)/|N|$; in particular $1 < \mathfrak{a}(G/N) < \mathfrak{a}(G)$. By induction, $\mathbb{F}[G/N]$ is not Lie centre-by-metabelian, and we are done.

So we may assume that A' is a minimal normal (2-)subgroup of G. Then $\Phi(A') = 1$, so A' is elementary abelian.

If $|A'| \geq 8$, then $A \subseteq C_G(A')$ by Lemma 0.2 (since A is not a counterexample to Theorem 1). In this case, A' may be regarded as an $\mathbb{F}_2[G/A]$-module. Let N be a simple submodule of A'. Then $N \cong Z_2$, since $G/A \cong Z_2$. But then $N \trianglelefteq G$ and $N < A'$ in contradiction to the minimality of A'.

Consequently $|A'| \leq 4$, in fact $|A'| = 4$, i.e. $A' \cong V_4$.

The action of G on A' gives a monomorphism φ: $G/C_G(A') \to \mathrm{Aut}(A') \cong S_3$. By 4.3, we may assume that φ is not surjective. If $|G : C_G(A')| \leq 2$, we again find a (trivial) simple submodule N of the $\mathbb{F}_2[G/C_G(A')]$-module A' in contradiction to the minimality of A'. Therefore $|G : C_G(A')| = 3$.

Then $G' \subseteq A \cap C_G(A') =: B$, so G/B is abelian, i.e. $G/B \cong Z_6$. We write $G = \langle g, B \rangle$. Then $g^6 \in B$, $g^3 \in C_G(A')$, $g^2 \in A$. We also may write $A' = \langle x, y \rangle$ with ${}^g x = y$ and ${}^g y = xy$. Then

$$(\mathbb{F}G)'' \ni [g + {}^x g, x + {}^g x] = [(1+xy)g, x+y] = (1+xy)(x+y+y+xy)g = (A')^+ g.$$

Clearly $G = \langle g, A \rangle$, and so $G' = (g, A)A'$. Since $G' > A'$, there is an element $a \in A$ with $(g, a) \notin A'$. But then $[a, (A')^+ g] = (A')^+ [a, g] = (A')^+ (1 + (g, a))ag \neq 0$. ∎

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